Bose-Einstein condensation of a coupled two-component Bose gas

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In experiments on excitons and spin-polarized atomic hydrogen, one deals with a gas involving two Bose branches (for example, paraexcitons and orthoexcitons). Based on a simple model for the coupling between these two components, we discuss how interconversion processes affect Bose condensation in such a system. Our analysis is based on an exact transformation to renormalized Bose particles which are uncoupled. We give results for the condensate fraction as a function of the temperature in the case when the two original Bose particles (a and b) are not in chemical equilibrium (i.e., the chemical potentials μ_a and μ_b are not equal) as well as in the case of chemical equilibrium ($\mu_a = \mu_b$). Our results are of interest in connection with current attempts to observe Bose condensation in atomiclike gases.

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I. INTRODUCTION

In this paper, we investigate Bose-Einstein condensation (BEC) in a simple model of a coupled two-component Bose gas. While we believe our results are of general interest in relation to recent attempts to find BEC in atomic gases (for reviews, see Ref. [1]), our specific motivation is to understand experiments done on a gas of optically-excited excitons in high quality Cu_2O crystals [2,3]. The lowest energy exciton in this system is split into two levels by the electron-hole exchange interaction: the spin singlet ($S = 0$) paraexciton is the lowest energy level and the spin triplet ($S = 1$) orthoexciton is 12 meV higher. Interconversion between these two exciton species involves a time that is much longer than the exciton lifetime due to electron-hole conversion. Our model consists of a two component Bose gas, which is assumed to be in thermal equilibrium but not necessarily in chemical equilibrium (i.e., the number of Bose particles of each species is determined by external conditions, such as pumping rate, etc.) For a review of current experimental evidence for BEC in the paraexciton branch of Cu_2O, we refer to the paper by Wolfe, Lin, and Snoke in Ref. [1]. For related problems in the context of the two lowest energy states of spin-polarized atomic hydrogen, see Refs. [4–6].

As a first step, we believe a detailed theoretical investigation of the simple model that we consider here is worthwhile. It is surprisingly rich in content and, as far as we are aware, this model has not been studied in the literature. In Cu_2O, this interconversion coupling involves the emission (ortho $\rightarrow$ para) or absorption (para $\rightarrow$ ortho) of an acoustic lattice phonon, which takes care of energy and momentum conservation. This more realistic model for interconversion is briefly discussed at the end of the paper.

II. MODEL HAMILTONIAN

Motivated by the coupled two-component exciton gas in Cu_2O, we introduce a simple model of a two-component Bose gas described by the grand canonical "Hamiltonian,"

$$\hat{K} = \hat{H} - \mu_a \hat{N}_a - \mu_b \hat{N}_b = \sum_k \left[E_k^a \hat{a}_k^\dagger \hat{a}_k + E_k^b \hat{b}_k^\dagger \hat{b}_k + M(\hat{a}_k^\dagger \hat{b}_k + \hat{b}_k^\dagger \hat{a}_k) \right], \quad (1)$$

where

$$E_k^a = \varepsilon_k - \mu_a, \qquad E_k^b = \varepsilon_k + 2\Delta - \mu_b. \quad (2)$$

This describes two Bose gas branches (lower one $\equiv a$ and upper one $\equiv b$, separated by an energy 2Δ). The center-of-mass kinetic energy is denoted by ε_k. The creation ($\hat{a}^\dagger, \hat{b}^\dagger$) and annihilation operators ($\hat{a}, \hat{b}$) of the two branches satisfy the usual Bose commutation relations,

$$\left[\hat{a}_k, \hat{a}_{k'}^\dagger\right]_- = \delta_{k,k'}, \qquad \left[b_k, b_{k'}^\dagger\right]_- = \delta_{k,k'}, \quad (3)$$

but commute between different branches. The coupling between a and b bosons is treated as a momentum-independent constant M for simplicity. The chemical potentials μ_a and μ_b are determined by the number of a and b bosons. We assume that our two component Bose gas is in thermal equilibrium (common temperature) due to interparticle collisions but not necessarily in chemical equilibrium (that is, μ_a may be different from μ_b).

The advantage of working with (1) is that it can be diagonalized in terms of new boson $\hat{c}$ and $\hat{d}$ operators. One finds that the appropriate transformation matrix is

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$$\begin{pmatrix} \hat{c}_k \\ \hat{d}_k \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{a}_k \\ \hat{b}_k \end{pmatrix} , \tag{4}$$

where

$$\left.\begin{matrix} \cos^2\theta \\ \sin^2\theta \end{matrix}\right\} = \frac{1}{2}\left(1 \pm \frac{\gamma'}{\Delta'}\right), \tag{5}$$

with

$$\Delta' \equiv \sqrt{\gamma'^2 + M^2} > 0 , \tag{6a}$$

$$\gamma' \equiv \Delta + \mu_- \tag{6b}$$

and

$$\mu_- = \frac{1}{2}(\mu_a - \mu_b) . \tag{7}$$

When diagonalized, the "Hamiltonian" is given by

$$\hat{K} = \sum_k (E_k^c \hat{c}_k^\dagger \hat{c}_k + E_k^d \hat{d}_k^\dagger \hat{d}_k) , \tag{8}$$

with the renormalized energies

$$\left.\begin{matrix} E_k^c \\ E_k^d \end{matrix}\right\} = \frac{E_k^a + E_k^b}{2} \mp \Delta' = \varepsilon_k + \Delta - \mu_+ \mp \Delta' , \tag{9}$$

where we have defined

$$\mu_+ \equiv \frac{1}{2}(\mu_a + \mu_b) . \tag{10}$$

In the above analysis, we have assumed for simplicity that $\varepsilon_k^a = \varepsilon_k^b \equiv \varepsilon_k$. In the more general case when $\varepsilon_k^a \neq \varepsilon_k^b$, we need to make the replacements,

$$\begin{aligned} &\varepsilon_k \Rightarrow \epsilon_k^a, \\ &\Delta \Rightarrow \Delta + \frac{1}{2}(\varepsilon_k^b - \varepsilon_k^a) \equiv \Delta_k . \end{aligned} \tag{11}$$

Strictly speaking, this generalization should be used for excitons since the paraexciton and orthoexciton masses are somewhat different (see Wolfe *et al.* in Ref. [1]).

We have reduced our coupled two-component Bose gas to a gas of two uncoupled bosons. In the limit of $M = 0$, the c bosons reduce to a bosons and the d bosons reduce to b bosons. However, when $M \neq 0$, the c- and d-boson operators involve linear combination of the a- and b-boson operators. It is useful to emphasize the difference between these new c and d bosons and the quasiparticles, which arise in the Bogoliubov theory of a weakly interacting one-component Bose gas [7,8]. While the latter theory is also based on an exact diagonalization, the Bogoliubov annihilation operator involves a superposition of *both* creation and annihilation operators of the original Bose particles. This is related to the fact that the Bogoliubov transformation makes use of the existence of a Bose condensate as a particle reservoir. As a result, the number of Bogoliubov quasiparticles is not fixed and thus there is no associated quasiparticle chemical potential. In contrast, the $\hat{c}$ and $\hat{d}$ annihilation operators in (4) only involve a superposition of annihilation operators ($\hat{a}$ and $\hat{b}$). The transformation (4) makes no use of the existence of a condensate reservoir and is, thus, valid at all temperatures.

As with the a and b bosons, the number of c and d bosons is fixed and one can introduce the associated chemical potentials μ_c and μ_d. If one diagonalizes the Hamiltonian $\hat{H}$ in (1), one obtains using (4)

$$\hat{H} = \sum_k (\bar{E}_k^c \hat{c}_k^\dagger \hat{c}_k + \bar{E}_k^d \hat{d}_k^\dagger \hat{d}_k) , \tag{12}$$

where

$$\left.\begin{matrix} \bar{E}_k^c \\ \bar{E}_k^d \end{matrix}\right\} = \epsilon_k + \Delta \mp \sqrt{\Delta^2 + M^2} . \tag{13}$$

Using this, we can rewrite (1) in the form

$$\hat{K} = \hat{H} - \mu_c \hat{N}_c - \mu_d \hat{N}_d , \tag{14}$$

where one easily finds

$$\left.\begin{matrix} \mu_c = \bar{E}_k^c - E_k^c \\ \mu_d = \bar{E}_k^d - E_k^d \end{matrix}\right\} = \sqrt{(\Delta + \mu_-)^2 + M^2} + \mu_+ \mp \sqrt{\Delta^2 + M^2} . \tag{15}$$

However, in this paper, we work in terms of E_k^c and E_k^d rather than $\bar{E}_k^c$ and $\bar{E}_k^d$.

With (8), we are dealing with two Bose gases which are uncoupled. The discussion of BEC is a straightforward extension of that for a one-component ideal Bose gas, discussed in all textbooks on statistical mechanics [7]. In particular, the momentum distribution of the new c and d bosons are given by the usual Bose distributions,

$$\begin{aligned} \langle \hat{n}_k^c \rangle &= \langle \hat{c}_k^\dagger \hat{c}_k \rangle = \frac{1}{e^{\beta E_k^c} - 1}, \\ \langle \hat{n}_k^d \rangle &= \langle \hat{d}_k^\dagger \hat{d}_k \rangle = \frac{1}{e^{\beta E_k^d} - 1} . \end{aligned} \tag{16}$$

We emphasize that the two Bose gases are in thermal equilibrium but not necessarily in chemical equilibrium.

We note that since $\Delta' > 0$, we have $E_k^c < E_k^d$ and so we only expect BEC of the c bosons. Let us examine the possibility of BEC in the zero center-of-mass momentum state $\mathbf{k} = \mathbf{0}$. Since we have

$$E_{k=0}^c = \Delta - \Delta' - \mu_+ , \tag{17}$$

we will have BEC for the c bosons when

$$\mu_+ = \Delta - \Delta' . \tag{18}$$

Since Δ and M are given constants, this condition is a relation [using (6a) and (6b)] between μ_+ and μ_-,

$$(\Delta - \mu_+)^2 = (\Delta + \mu_-)^2 + M^2 . \tag{19}$$

This is equivalent to a relation between $\mu_a (= \mu_+ + \mu_-)$ and $\mu_b (= \mu_+ - \mu_-)$. Assuming (18) holds, the renormalized boson excitation energies in (9) reduce to

$$\begin{aligned} E_k^c &= \varepsilon_k, \\ E_k^d &= \varepsilon_k + 2\Delta', \end{aligned} \tag{20}$$

in the Bose-condensed state of the c bosons (i.e., at $T < T_c$).

In order to determine μ_+ and μ_-, we use the formulas

$$N_a = \sum_k \langle \hat{a}_k^\dagger \hat{a}_k \rangle,$$
$$N_b = \sum_k \langle \hat{b}_k^\dagger \hat{b}_k \rangle \ , \tag{21}$$

which implicitly give the number of a and b bosons in terms of the chemical potentials μ_a and μ_b. Using the normalized boson representation, (21) can be rewritten as

$$N_a = (\sin^2\theta) N_0^c + \sum_{k\neq 0} \left[\cos^2\theta \langle \hat{c}_k^\dagger \hat{c}_k \rangle + \sin^2\theta \langle \hat{d}_k^\dagger \hat{d}_k \rangle \right],$$
$$N_b = (\cos^2\theta) N_0^c + \sum_{k\neq 0} \left[\sin^2\theta \langle \hat{c}_k^\dagger \hat{c}_k \rangle + \cos^2\theta \langle \hat{d}_k^\dagger \hat{d}_k \rangle \right] \ , \tag{22}$$

where N_0^c is the number of c bosons which are in the $\mathbf{k} = \mathbf{0}$ state. The expression in (22) shows explicitly that while only the lower energy c bosons are condensed, this implies BEC of both a and b bosons. Making use of (5), (16), and (20), in (22), we obtain for $T \leq T_c$,

$$N_a + N_b = N_0^c + \sum_{\mathbf{k}\neq\mathbf{0}} \left[\frac{1}{e^{\beta\varepsilon_k} - 1} + \frac{1}{e^{\beta(\varepsilon_k + 2\Delta')} - 1} \right], \tag{23a}$$

$$N_a - N_b = \frac{\gamma'}{\Delta'} N_0^c - \frac{\gamma'}{\Delta'} \sum_{\mathbf{k}\neq\mathbf{0}} \left[\frac{1}{e^{\beta\varepsilon_k} - 1} - \frac{1}{e^{\beta(\varepsilon_k + 2\Delta')} - 1} \right] . \tag{23b}$$

The two formulas (23a) and (23b) are the basis of our analysis in Sec. III. Recalling from (6a) that γ' is a known function of Δ' and M, the coupled equations (23a) and (23b) uniquely determine the value of Δ' and N_0^c, for given values of M, T, N_a, and N_b. Perhaps surprisingly, the values for Δ' and N_0^c given by (23a) and (23b) are seen to be *independent* of the initial energy level splitting 2Δ between the a and b bose branches. In turn, the new energy level difference $2\Delta'$ [see (9)] is also independent of the original level difference 2Δ. This feature gives us more confidence in our simple model for the coupling as defined in (1), which does not take into account energy conservation during the interconversion process. Once we have the value of Δ', we can determine μ_+ and μ_- using (6b) and (18). Clearly the values of the chemical potentials μ_a and μ_b depend very much on the value of the energy splitting 2Δ.

III. NUMERICAL RESULTS

In this section, we present some explicit results based on the coupled equations (23a) and (23b), which can be expressed in terms of a standard Bose-Einstein integral [7],

$$\frac{1}{\lambda_T^3} F_{3/2}(\alpha) = \frac{1}{V} \sum_{\mathbf{k}\neq\mathbf{0}} \frac{1}{e^{\beta\varepsilon_k + \alpha} - 1} \ , \tag{24}$$

where the thermal de Broglie wavelength is $\lambda_T \equiv (2\pi\hbar^2/mk_BT)^{1/2}$. $F_{3/2}(\alpha)$ is a decreasing function of α, with

$$F_{3/2}(\alpha) \leq F_{3/2}(\alpha = 0) = 2.612 \ . \tag{25}$$

One obtains

$$n_a + n_b = n_0^c + \frac{1}{\lambda_T^3} \left[F_{3/2}(2\beta\Delta') + 2.612 \right], \tag{26}$$

$$n_a - n_b = \frac{\gamma'}{\Delta'} n_0^c - \frac{\gamma'}{\Delta'} \frac{1}{\lambda_T^3} \left[F_{3/2}(2\beta\Delta') - 2.612 \right] \ , \tag{27}$$

where $\Delta' \equiv \sqrt{\gamma'^2 + M^2}$ and $n_a = N_a/V$, etc. In recent experiments on optically-excited Cu_2O, it is generally believed that the paraexcitons and orthoexcitons are not in chemical equilibrium (see the review paper by Wolfe *et al.* in Ref. [1]). In this case, the values of n_a and n_b in (26) and (27) are externally determined and thus $\mu_a \neq \mu_b$, as mentioned in the last paragraph of Sec. II. In Fig. 1, we plot the BEC transition temperature T_c (at which n_0^c vanishes) as a function of the density n_a for given values of M and n_b. We chose $M = 0.6$ meV and n_a, n_b in the range of $10^{17} - 10^{19}$ cm^{-3}, as being appropriate in experiments on Cu_2O [1–3]. Figure 1 shows that for a fixed density of a bosons, T_c increases as the density of b bosons increases. This implies that the presence of b bosons and the interconversion between b and a bosons make it "easier" for the system to Bose condense. In a two-component exciton gas in Cu_2O, the BEC line (T_c vs n_a) for the paraexcitons lies higher than that for an ideal Bose gas of density n_a.

The condensate density below T_c is also modified by the interconversion interaction M. In Fig. 2, we plot the density of a and b bosons in the condensate as a function of the temperature T, with comparison to the case of an ideal gas. As we have noted earlier, both a and b bosons will Bose condense, while only the low energy c bosons

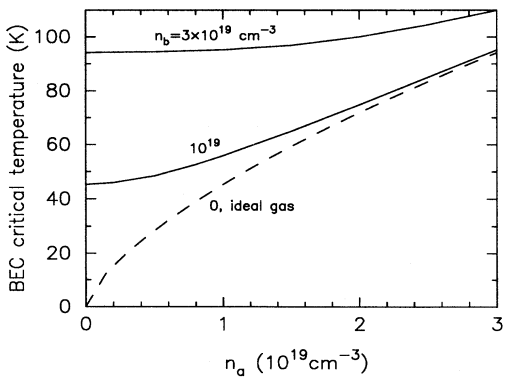

FIG. 1. Bose-Einstein condensation transition temperature vs density of a bosons, for different values of the density n_b of the b bosons. At fixed n_a, T_c increases with n_b.

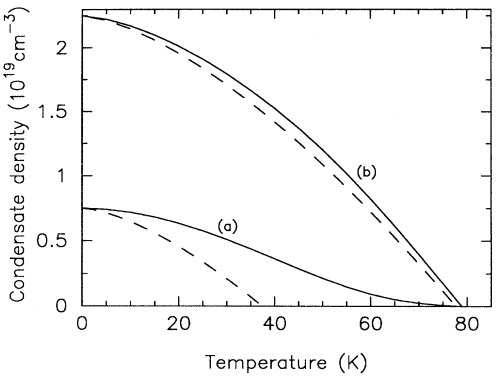

FIG. 2. Condensate density vs temperature for both a and b bosons. We choose $n_b = 3n_a = 2.25 \times 10^{19}$ cm^{-3}. Bose-Einstein condensation occurs at $T_c = 79$ K for both components in the presence of interconversion. For the corresponding uncoupled ideal gases (dashed lines) $T_c^a = 37$ K and $T_c^b = 78$ K. Note that all the bosons are condensed into the zero-momentum state at $T = 0$.

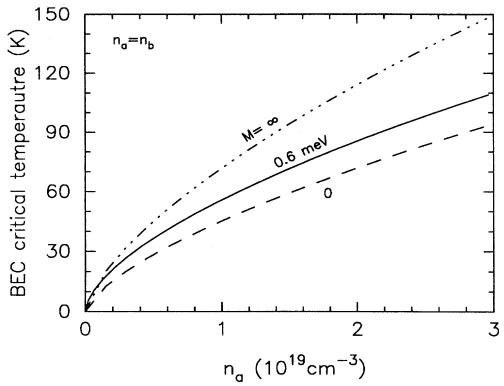

FIG. 3. T_c vs n_a for different values of the interconversion coupling strength M. These results are for the special case $n_a = n_b$, where the coupling has the strongest effect. The BEC transition temperature increases with the coupling strength M.

do so. In the language of excitons in Cu_2O, we have

$$n_0^{\rm para} = \frac{1}{2}\left(1 + \frac{\gamma'}{\Delta'}\right) n_0^c,$$
$$n_0^{\rm ortho} = \frac{1}{2}\left(1 - \frac{\gamma'}{\Delta'}\right) n_0^c\ , \qquad (28)$$

with $n_0^{\rm para} + n_0^{\rm ortho} = n_0^c$. Bose condensation in both branches occurs at the same temperature, even though the densities of the two species are different. In contrast, if a and b bosons are independent of each other ($M = 0$), BEC occurs at different temperatures (if $n_a \neq n_b$).

Another interesting feature shown by Fig. 2 is that at $T = 0$, all the a and b bosons are condensed into the zero-momentum state, which is exactly what happens in an ideal one-component Bose gas. As is well known, in a one-component interacting Bose gas, a finite number of particles are removed from the condensate even at $T = 0$ [7]. Thus, we see that the consequences of interconversion and interparticle interactions on BEC are different.

To examine the effect of the coupling strength M on BEC, in Fig. 3 we plot T_c as a function of n_a for different values of M. As M varies from zero to "infinity" ($2\beta M \gg 1$), the transition temperature increases for fixed values of n_a. In other words, as the interconversion process become stronger, BEC becomes "easier" in the system. In the limit of very large M, i.e., very strong coupling, the BEC transition line becomes very close to that of a one-component ideal gas with a particle density equal to the *combined* particle density of the two components. In this case, the interconversion process "blurs" the distinction between a and b particles. A plot of the effective condensate density ($n_0^c \equiv n_0^a + n_0^b$) versus temperature at different values of M (Fig. 4) also indicates the "positive" effect of interconversion interactions on BEC. The curve corresponding to $M = 0$ describes two uncoupled gases of the same density which Bose condense at the same temperature (hence $n_a^0 = n_b^0$).

Finally, for completeness, we discuss the chemical potentials μ_a and μ_b both below and above T_c. For this we need a more general version of (26) and (27) valid for T above T_c,

$$n_a + n_b = F_{3/2}(\alpha_-) + F_{3/2}(\alpha_+),$$
$$n_a - n_b = \frac{\gamma'}{\Delta'}[F_{3/2}(\alpha_-) - F_{3/2}(\alpha_+)]\ , \qquad (29)$$

where we have defined

$$\alpha_\mp \equiv \beta(\Delta - \mu_+ \mp \Delta'), \qquad (30)$$

and the BE integral is given by (24). If we assume $n_a < n_b$, inspection of (29) shows that we must have $\gamma' < 0$ [since $\alpha_- < \alpha_+$, we have $F_{3/2}(\alpha_-) > F_{3/2}(\alpha_+)$] . For given values of n_a, n_b, M, and T, one can solve (29) for

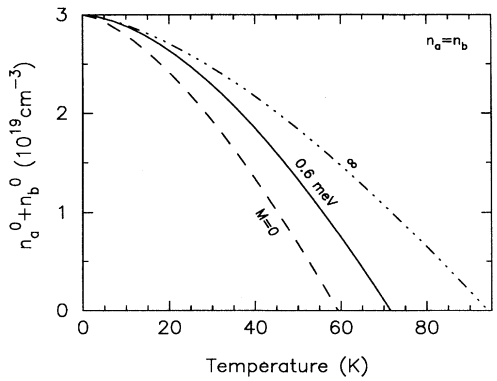

FIG. 4. The total condensate density ($n_c^0 = n_a^0 + n_b^0$) vs temperature, for different values of M. We take $n_a = n_b = 1.5 \times 10^{19}$ cm^{-3}. The condensate fraction at a given temperature increases with M.

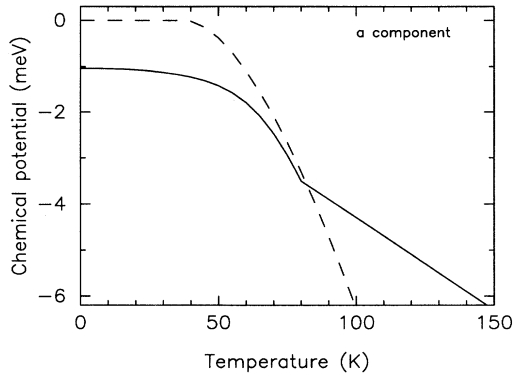

FIG. 5. Chemical potential μ_a vs T for a bosons in a two-component Bose gas, for $n_b = 3n_a = 2.25 \times 10^{19}$ cm^{-3}. For comparison, we show an ideal Bose gas of density n_a (dashed line), where $T_c = 37$ K. A kink appears at the transition temperature $T_c = 79$ K in the coupled system.

Δ' and μ_+. Using (6a) and (6b), μ_- is given by

$$\mu_- = -\sqrt{\Delta'^2 - M^2} - \Delta \ , \tag{31}$$

for $n_a < n_b$. Knowing $\mu_\pm$, we can find μ_a and μ_b.

In Fig. 5, we plot the temperature dependence of the chemical potential $\mu_a(T)$ for the a bosons, for $2\Delta = 12$ meV and $n_b = 3n_a = 2.25 \times 10^{19}$ cm^{-3}. The "kink" appears at the BEC transition temperature $T_c = 78$ K of the coupled system. For comparison, we plot the equivalent result for a one-component ideal Bose gas of density n_a (in this case, $\mu_a = 0$ for $T < T_{\rm BEC} = 37$ K). In Fig. 6, we give the analogous results for $\mu_b(T)$. One sees that, in contrast to an ideal Bose gas, μ_a and μ_b both vary with temperature in the Bose condensed phase. We also note that (see discussion at the end of Sec. II) μ_a and μ_b depend very explicitly on the magnitude of the assumed energy level splitting 2Δ, in contrast to the results in Figs. 1–4.

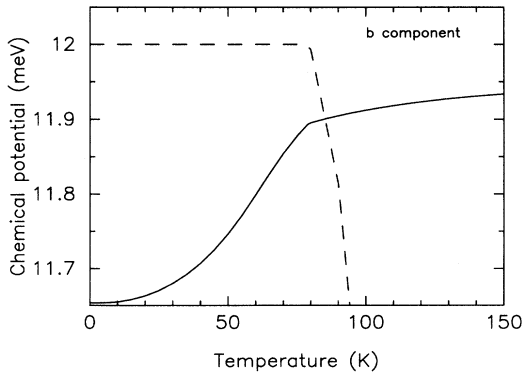

FIG. 6. Chemical potential μ_b vs T for b bosons, for the same parameters as in Fig. 5. For comparison, we show an ideal Bose gas of density n_b (dashed line) where $T_c = 78$ K. The kink is again at 79 K, the BEC transition temperature of the coupled system (see Fig. 2).

Whether the two Bose branches are in chemical equilibrium or not depends on the rate of interconversion (see also Refs. [4–6]). If this rate is rapid enough, then one has chemical equilibrium defined by the condition $\mu_a = \mu_b \equiv \mu$ [4]. In this case, $\mu_- = 0$ and $\mu_+ = \mu$. From (6a), (6b), and (18), this means $\gamma' = \Delta$ and $\Delta' = \sqrt{\Delta^2 + M^2} = \Delta - \mu$, and thus the common chemical potential is determined by ($T \leq T_c$)

$$\mu = \Delta - \sqrt{\Delta^2 + M^2} \ . \tag{32}$$

In this case, Eqs. (26) and (27) should be viewed as determining the value of n_0^c and $n_a - n_b$ for a given value of $n_a + n_b \equiv n$ (for fixed values of M and T). More specifically, (26) gives

$$n_0^c = n - \frac{1}{\lambda_T^3}\left[F_{3/2}(2\beta\sqrt{\Delta^2 + M^2}) + 2.612\right] \ , \tag{33}$$

while (27) reduces to

$$\begin{aligned} n_a - n_b &= \frac{\Delta}{\Delta'} n_0^c - \frac{\Delta}{\Delta'}\frac{1}{\lambda_T^3}\left[F_{3/2}(2\beta\sqrt{\Delta^2 + M^2}) - 2.612\right] \\ &= \frac{\Delta}{\Delta'}\left[n - \frac{2F_{3/2}}{\lambda_T^3}(2\beta\sqrt{\Delta^2 + M^2})\right] \ . \end{aligned} \tag{34}$$

The BEC transition temperature is where n_0^c in (33) vanishes. For excitons with the level splitting $2\Delta = 12$ meV, the requirement of chemical equilibrium results in almost all the particles being paraexcitons (the lower a state) and very few being orthoexcitons (the upper b state). As a result, the features related to BEC are almost unchanged from those of an ideal Bose gas composed of paraexcitons.

IV. CONCLUSIONS

In this paper, we have given a careful analysis of a two-component Bose gas in which a simple mechanism for interconversion is allowed. It gives considerable insight into the physics involved in BEC with two components. In a sense, our model calculation is the simplest generalization of the original work by Einstein in 1925. The problem can be reduced to a gas of two noninteracting Bose gases with a renormalized energy spectrum. The result is that while BEC only appears in the lowest renormalized Bose branch, this corresponds to BEC of both of the original Bose species. The results in Figs. 1–4 show how the interconversion interaction affects BEC. In a coupled two-component Bose gas such as excitons in Cu_2O, we find the paraexcitons (the lowest energy state) will start to Bose condense at a higher transition temperature than the equivalent ideal gas does, as a result of the presence of interconversion between orthoexcitons and paraexcitons. This effect is opposite to that of two-particle collisions. The present results are of interest in the current experimental search for BEC in the two-component exciton gas in optically-excited Cu_2O [1–3]. They should be relevant to spin-polarized atomic hydrogen, which also involves two low-lying Bose branches [4–6].

There are several important generalizations which are needed to make the present work more relevant to experiments in Cu_2O. We have ignored exciton-exciton two-particle interactions [8], which would make our problem into an interacting two-component Bose gas. In addition, our model Hamiltonian should also really include the acoustic lattice phonons involved in the interconversion between orthoexcitons and paraexcitons in Cu_2O [1]. This is a more difficult problem since it leads to a hybridization of the lattice acoustic phonons and the exciton branches. However, it is useful to briefly relate the results of our simple model (1) to those based on such a coupled exciton–acoustic-phonon system. The latter problem is best dealt with thermal Green's functions [8], in which one includes the phonon-induced interconversion through its effect on the exciton self-energy. The a-exciton self-energy corresponding to one phonon exchange can be shown to be given by the following expression [9]:

$$\Sigma_a(\mathbf{k},\omega) = \sum_{\mathbf{q}} |M_q|^2 \frac{N(\omega_q) - N(E^b_{k-q})}{\omega - (E^b_{k-q} - \omega_q)} , \tag{35}$$

where $\omega_q = cq$ is the acoustic-phonon energy, $N(E)$ is the Bose distribution, and E^b_k is defined in (2). For comparison, one may verify starting with (1) that the a-exciton Green's function,

$$G_a(\mathbf{k},\omega) = \frac{1}{\omega - E^a_k - \Sigma_a(k,\omega)} , \tag{36}$$

has the following exact self-energy:

$$\Sigma_a(k,\omega) = \frac{|M|^2}{\omega - E^b_k} . \tag{37}$$

It is easy to check that the poles of $G_a(\mathbf{k},\omega)$ are given by E^c_k and E^d_k in (9).

Comparing (35) and (37), we see that when the coupling to the lattice phonon is included, the exciton self-energy depends on the occupancy of the exciton states as well as the number of phonons present. This introduces a much more nonlinear aspect to the coupled-exciton problem, with new physics. In a future paper, we will use (35) and (36) (and the analogous expressions for the b boson, in which a phonon is emitted) as the basis for a more realistic study of BEC in a gas of orthoexciton and paraexcitons [9].

Finally, we emphasize that throughout our discussion, we have assumed that the densities of a and b bosons (n_a and n_b) are determined externally (or are appropriate to chemical equilibrium). A much more ambitious problem is to work with the kinetic equations, which describe the time-dependent values of n_a and n_b in the optically-pumped exciton experiments of Refs. [1–3].

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